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Conformal invariance and integrable models

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Abstract. Usually, an integrable nonlinear partial differential equation can be transformed to its conformal invariant form (Schwartz form). Using the conformal invariance of the integrable models, we can obtain many interesting results. In this paper, we will focus mainly in obtaining new symmetries and new integrable models. Starting from the conformal invariance of an integrable model, one can obtain infinitely many non-local symmetries. Many types of $(1+1)$- and $(2+1)$-dimensional new sine–Gordon (or sinh–Gordon) extensions are obtained from the conformal flow equations of the Koerteweg–de Vries type equations. Many other kinds of integrable models can be obtained from the conformal constraints of the known integrable models.

1. Introduction

Since Gardner, Greene, Kruskal and Miura solved the Korteweg–de Vries (KdV) equation by means of the inverse scattering transformation (IST) approach in 1967 [1], the modern theory of solitons has been widely applied in physics and deeply studied in mathematics. A wealth of interesting properties of soliton equations have been revealed. For example, a soliton equation possesses N-soliton solutions, Bäcklund and Darboux transformations, the Lax pair, a bilinear form, a zero-curvature form, the Painlevé property, infinitely many symmetries and conservation laws [2] etc.

In this paper, I will show that many interesting results of integrable models are related to the conformal invariance. For instance, infinitely many non-local (and local) symmetries of integrable models can be obtained from the conformal invariance of its Schwartz form. Starting from the conformal flow equations of the KdV type equations, many kinds of sinh–Gordon (shG) (and/or sine–Gordon (SG)) extensions can be obtained. Using the conformal symmetry constraints, we can see that many other types of integrable models are linked to each other.

In the next section, we show that infinitely many non-local (and then local) symmetries of the KdV equation can be obtained from the single conformal invariance of its Schwartz form. In section 3, it is shown that the well known SG (or ShG) equation is just a variant of the conformal flow equation of the KdV equation. On the other hand, starting from the conformal flows of the KdV extensions, many kinds of ShG (or SG) extensions both in $(1+1)$ and $(2+1)$ dimensions are also given in section 3. In section 4, one can see that using the symmetry constraints related to the conformal invariance of the KP

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equation, the $(1+1)$-dimensional AKNS system and a special type of $(2+1)$-dimensional AKNS system (including the asymmetric Davey–Stewartson (DS) equation and asymmetric Nizhnik–Novikov–Veselov (NNV) system) can be obtained. The last section is a summary and discussion.

2. Infinitely many symmetries of the KdV equation from the conformal invariance

Mathematically, for a nonlinear partial differential equation, say, KdV equation, there must be *infinitely many* symmetries (and then conservation laws) to guarantee its integrability. In this section we write down the concrete procedure to derive the infinitely many symmetries of the KdV equation from the conformal invariance of its Schwartz form.

A symmetry of the KdV equation,

$$u_t = 6uu_x + u_{xxx} \equiv K(u) \tag{1}$$

is defined as a solution of its linearized equation

$$\sigma_t^u = 6\partial_x(u\sigma^u) + \partial_x^3\sigma^u = \lim_{\epsilon\to 0}\frac{\partial}{\partial\epsilon}K(u+\epsilon\sigma^u) \equiv K'\sigma^u. \tag{2}$$

In other words, equation (1) is form invariant under the transformation

$$u \longrightarrow u + \epsilon\sigma^u \tag{3}$$

with an infinitesimal parameter ϵ.

Using the transformation

$$u = \lambda - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)_x - \frac{1}{4}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 \tag{4}$$

the KdV equation (1) can be changed to its Schwartz form (the SKdV equation)

$$\phi_t = \{\phi; x\}\phi_x + 6\lambda\phi_x \tag{5}$$

where $\{\phi; x\} = (\phi_{xxx}/\phi_x) - \frac{3}{2}(\phi_{xx}/\phi_x)^2$ is the Schwartz derivative. The arbitrary constant λ has been entered into equations (4) and (5) because of Galilean invariance of the KdV equation (1). From equation (4), we know that the symmetries of the KdV and SKdV equations are related to each other by

$$\sigma^u = -\frac{1}{2}\partial_x\left(\frac{1}{\phi_x}\partial_x^2 - \frac{\phi_{xx}}{\phi_x^2}\partial_x\right)\sigma^\phi - \frac{1}{2}\left(\frac{\phi_{xx}}{\phi_x}\right)\left(\frac{1}{\phi_x}\partial_x^2 - \frac{\phi_{xx}}{\phi_x^2}\partial_x\right)\sigma^\phi. \tag{6}$$

It is well known that the SKdV equation (5) is invariant under the finite Möbious (conformal) transformation:

$$\phi \longrightarrow \frac{a+b\phi}{c+d\phi} \qquad (ad \neq cb). \tag{7}$$

The finite conformal transformation (7) can be considered as a combination of three transformations: translation–inverse–translation. Here we only treat a special case for $a = 0$, $b = c = 1$ and $d = \epsilon$. In this special case, the conformal transformation (7) can be written to its infinitesimal form:

$$\phi \longrightarrow \phi - \epsilon\phi^2 \tag{8}$$

i.e. $-\phi^2$ is a symmetry of the SKdV equation (5). Using the transformation relation (6), we get a non-local symmetry of the KdV equation corresponding to the conformal invariance of the SKdV equation

$$\sigma^u = 2\phi_{xx} = \partial_x\psi^2 \equiv K_0^{(1)}(\lambda) \tag{9}$$

where ψ and u are related by the well known Schrödinger equation

$$\psi_{xx} + u\psi = \lambda\psi. \tag{10}$$

Now starting from the single non-local symmetry $K_0^{(1)}$, we can obtain infinitely many non-local symmetries of the KdV equation. Because of the parameter λ being an arbitrary constant, we can treat it as a small parameter and expand $K_0^{(1)}(\lambda)$ as a series in λ:

$$K_0^{(1)}(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\partial^n}{\partial\lambda^n} K_0^{(1)}(\lambda) \right) \Bigg|_{\lambda=0} \lambda^n. \tag{11}$$

Substituting equation (11) into the symmetry definition equation (2), we can conclude that

$$K_n^{(1)} \equiv \frac{1}{n!} \frac{\partial^n}{\partial\lambda^n} K_0^{(1)}(\lambda)|_{\lambda=0}$$

must also be a symmetry of the KdV equation for all $n = 0, 1, 2, \ldots$ because of the linearity of equation (2) and λ being arbitrary. That is to say, the conformal invariance of the KdV equation (and its variants) implies that it possesses a set of infinitely many symmetries $K_n^{(1)}$. To give an explicit form of $K_n^{(1)}$, we solve equation (10) formally. The result reads $(\psi(\lambda = 0) \equiv \psi_0)$

$$\psi(\lambda) = \sum_{k=0}^{\infty} (\partial_x^2 + u)^{-k} \psi_0 \lambda^k = \sum_{k=0}^{\infty} (\psi_0 \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0)^k \psi_0 \lambda^k. \tag{12}$$

Substituting equation (9) with (12) into (11) yields

$$K_n^{(1)} = 2 \sum_{k=0}^{n} ((\psi_0 \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0)^k \psi_0)((\psi_0 \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0)^{n-k} \psi_0)_x \tag{13a}$$

$$= 2^{2n} (\partial_x \psi_0^2 \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0^2)^n (\psi_0^2)_x \equiv 2^{2n} \Phi_u^{-n} (\psi_0^2)_x \tag{13b}$$

where Φ_u, the recursion operator of the KdV equation, appears naturally when we try to write (13a) as a single term. The set of infinitely many symmetries $K_n^{(1)}$ is just one set of known non-local symmetries obtained by other approaches [3, 4].

The other two sets of infinitely many non-local symmetries

$$K_n^{(2)} = 2^{2n} \Phi_u^{1-n} (\psi_0^2 \partial_x^{-1} \psi_0^{-2} \partial_x^{-1} \psi_0^{-2})_x \tag{14}$$

$$K_n^{(3)} = 2^{2n} \Phi_u^{-n} (\psi_0^2 \partial_x^{-1} \psi_0^{-2})_x \tag{15}$$

given in [3] can be re-obtained easily from $K_n^{(1)}$ because if ψ is a solution of the Schrödinger equation (10), then

$$c_1\psi + c_2\psi \partial_x^{-1} \psi^{-2} \tag{16}$$

is also a solution of (10).

On the other hand, there is a set of infinitely many local symmetries

$$K_n^{(0)} = \Phi_u^n u_x \tag{17}$$

of the KdV equation (1). Actually, the existence of the set of non-local symmetries $K_n^{(1)}$ implies the existence of the set of local symmetries $K_n^{(0)}$ [5] because of the flow equation of the KdV (1) corresponding to the conformal invariance can be transformed to the well known sinh–Gordon (ShG) equation

$$w_{xt} = \sinh(2w) \tag{18}$$

and the ShG equation is symmetric for the spacetime $\{x, t\}$.

In summary, for the KdV equation the conformal invariance implies the existence of one set of infinitely many local symmetries and three sets of non-local symmetries.

3. Conformal flows of the KdV-type equations and the ShG extensions

For the KdV equation, one can easily prove that the well known $(1+1)$-dimensional ShG (or sine–Gordon (SG)) equation (18) is just a variant form of the conformal flow equation of the KdV equation (1)

$$u_t = \phi_{xx} = (\psi^2)_x. \tag{18$'$}$$

Actually the conformal flow equation (18)$'$ with (10) and the ShG equation (18) are related by the Miura transformation (4) with $\phi_x = \exp w$.

On the other hand, there are many $(1+1)$-dimensional and $(2+1)$-dimensional KdV-type integrable extensions. We believe that for every integrable KdV extension, there is a corresponding ShG (or SG) extension. The most convenient method to obtain the corresponding ShG (or SG) extension is to study the conformal flow equation of the KdV extension. In this section, we give out some concrete $(1+1)$- and $(2+1)$-dimensional ShG extensions from the conformal flow equations of the KdV extensions.

3.1. Coupled ShG extension from the Hirota–Satsuma model

In $(1+1)$-dimensions one of the most famous KdV extensions is the so-called Hirota–Satsuma (HS) system [6]:

$$u_t = 6uu_x + u_{xxx} - 6vv_x \tag{19}$$

$$v_t = -6uv_x - 2v_{xxx}. \tag{20}$$

It is obvious that the HS system reduces back to the KdV equation (1) for $v = 0$.

To find the corresponding ShG extension related to the HS system, we should first transform the HS system (19) and (20) to its conformal invariant form. The most convenient way to obtain the conformal invariant form of the HS system is to use the standard singularity analysis method [7].

Using the truncated singularity analysis, we obtain the conformal invariant form of the HS system

$$\phi_{xxx}\left\{\frac{4\phi_x}{\psi}\left[\left(\frac{\phi_t}{\phi_x} - \{\phi; x\}\right)_x + \frac{3\phi_{xx}\psi^2}{2\phi_x^3}\right]\right\}_x = \left[2\psi_{xxx} + \psi_x\left(\frac{\phi_t}{\phi_x} - 4\{\phi; x\} - 3\frac{\phi_{xx}}{\phi_x^2} - \frac{3\psi^2}{2\phi_x^2}\right) + \psi_t\right] \tag{21}$$

$$\frac{2\phi_x^2}{3}\left(2\psi - \frac{\phi_x\psi_x}{\phi_{xxx}}\right)\left(\frac{\phi_t}{\phi_x} - 4\{\phi; x\} - 3\frac{\phi_{xx}}{\phi_x^2} - \frac{3\psi^2}{2\phi_x^2}\right) - \frac{2\phi_x^3}{3\phi_{xxx}}(2\psi_{xxx} + \psi_t) + 4\phi_x(\psi\phi_{xxx} + \phi_x\psi_{xx} - \psi_x\phi_{xx}) - 2\psi\phi_{xx}^2 + \psi^3 = 0 \tag{22}$$

where $\{\phi, \psi\}$ equations (21) and (22) and $\{u, v\}$ equations (19) and (20) are related by

$$8\phi_x^3v_x - 4\phi_x\phi_{xx}\psi_x + 8u\psi\phi_x^2 + 4\phi_x^2\psi_{xx} + 4\psi\phi_x\phi_{xxx} - 2\psi\phi_{xx}^2 + \psi^3 = 0 \tag{23}$$

$$-4\phi_x\phi_{xx}\phi_{xxx} + 2\psi v\phi_x^2 + 4u_x\phi_x^3 + 2\psi\psi_x\phi_x + 2\phi_{xx}^3 - \phi_{xx}\psi^2 + 2\phi_x^2\phi_{xxxx} = 0. \tag{24}$$

Obviously equation (21) for $\psi = 0$ is just the usual Schwartz KdV equation.

The conformal flow equation of the HS system reads

$$u_t = 2\phi_{xx} \tag{25}$$

$$v_t = \psi \tag{26}$$

with equations (23) and (24). When $\psi = 0$ and $v = 0$, equation (24) is just the Miura transformation

$$u = -w_{xx} - w_x^2 + \lambda \tag{27}$$

with $\phi_x = \exp(2w)$ and equation (25) is just the ShG equation (18). For the general $v \neq 0$, using the extended Miura transformation

$$u = -w_{xx} - w_x^2 - \tfrac{1}{2} f \tag{28}$$

we get a coupled integrable ShG extension related to the HS system:

$$w_{xt} = 2\sinh(2w) + 2\sinh(2F) \tag{29}$$

$$f_t = 4(\partial_x + 2w_x)\sinh(2F) \tag{30}$$

$$f_x = (v_t v_{xt} - w_x v_t^2)\exp(-4w) + v v_t \exp(-2w) \tag{31}$$

$$8\exp(2w)v_x + 4(\partial_x - 2w_x)v_{xt} - f v_t + v_t^3 \exp(-4w) = 0. \tag{32}$$

3.2. (2 + 1)-dimensional ShG extensions from (2 + 1)-dimensional KdV-type equations

In addition to the (1 + 1)-dimensional KdV extensions, there are some different (2 + 1)-dimensional KdV extensions such as the KP equation, the Boiti–Leon–Manna–Pempinelli (BLMP) equation, the breaking soliton equation and Nizhnik–Novikov–Veselov (NNV) equation. In this subsection, we transform the conformal flow equations of two (2 + 1)-dimensional KdV-type equations to the (2 + 1)-dimensional ShG extensions.

3.2.1. ShG equation from the KP equation. Using the truncated singularity analysis on the KP equation [8]

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0 \tag{33}$$

we get its conformal invariant form (Schwartz form)

$$\left(\frac{\phi_t}{\phi_x} + \{\phi;\ x\} + \frac{3\phi_y^2}{2\phi_x^2}\right)_x + 3\left(\frac{\phi_y}{\phi_x}\right)_y = 0. \tag{34}$$

The KP equation (33) and its Schwartz form (34) are related by

$$u = -\frac{1}{2}\{\phi;\ x\} - \frac{\phi_{xx}^2}{2\phi_x^2} - \frac{\phi_y^2}{4\phi_x^2} + \frac{1}{2}\int^x \left(\frac{\phi_y}{\phi_x}\right)_y \mathrm{d}x. \tag{35}$$

From equation (35) we know that the non-local symmetry

$$\sigma = 2\phi_{xx} \equiv (\psi\psi^*)_x \tag{36}$$

of the KP equation (33) corresponds to the conformal invariance (Möbious transformation invariance) of the Schwartz KP equation (34), where ψ and ψ^* are just the spectral functions of the 'time-y' dependent Schrödinger equation

$$\psi_y + (\partial_x^2 + u - \lambda)\psi = 0 \tag{37}$$

and its adjoint

$$-\psi_y^* + (\partial_x^2 + u - \lambda)\psi^* = 0 \tag{38}$$

respectively.

The conformal flow equation of the KP equation (33) now reads

$$u_t = (\psi\psi^*)_x \tag{39}$$

with (37) and (38).

Using the two-dimensional Miura transformation,

$$u = -w_{xx} - w_y - w_x^2 \tag{40}$$

the conformal flow of the KP equation will be changed to a (2+1)-dimensional sinh–Gordon extension:

$$(w_y + w_{xx} + w_x^2)_{yt} = (s_x\, \mathrm{e}^{2w})_{xx} \tag{41}$$

$$(2w_x + \partial_x)(w_{xt} - \tfrac{1}{2}C_1\, \mathrm{e}^{2w} + \tfrac{1}{2}C_2\, \mathrm{e}^{-2w}) + w_{yt} + (s\, \mathrm{e}^{2w})_x = 0. \tag{42}$$

The differential equation system (41) and (42) is equivalent to the complicated integro-differential equation (23) of [9]. It is obvious that the $(2+1)$-dimensional ShG extension (41) and (42) will reduce back to the known $(1+1)$-dimensional ShG equation for $s = 0$ and $w_y = 0$.

The same Miura transformation (40) will change the whole KP hierarchy and negative KP hierarchy to the potential modified KP hierarchy and $(2+1)$-dimensional sinh–Gordon hierarchy, respectively [10].

3.2.2. ShG equations from the BLMP and NNV equations. Another known $(2 + 1)$-dimensional KdV extension is the so-called BLMP equation [11]

$$u_t + u_{xxx} = 3(u\partial_y^{-1}u_x)_x \tag{43}$$

which can be considered as the space $\{x, y\}$ asymmetric form of the NNV equation [12]

$$u_t + u_{xxx} + u_{yyy} - 3(u\partial_y^{-1}u_x)_x - 3(u\partial_x^{-1}u_y)_y = 0. \tag{44}$$

From the singularity analysis of the BLMP equation (43), we can write the conformal invariant form of (43) in the form

$$\begin{aligned}\left(\partial_y + \frac{\phi_y}{\phi_x}\partial_x - \left(\frac{\phi_y}{\phi_x}\right)_{xy} - \frac{\phi_y}{\phi_x}\left(\frac{\phi_y}{\phi_x}\right)_{xx} + 2\left(\left(\frac{\phi_y}{\phi_x}\right)_x\right)^2\right)&\\ \times\left(\frac{\phi_t}{\phi_x} + \{\phi; x\}\right)_y - 3\{\phi; x\}_y\left(\left(\frac{\phi_y}{\phi_x}\right)_x\right)^2 = 0.&\end{aligned} \tag{45}$$

The BLMP equation (43) and its Schwartz form (45) now are related by

$$\begin{aligned}u(2\phi_y\phi_x\phi_{xy} - \phi_y^2\phi_{xx} - \phi_x^2\phi_{yy}) + \phi_y\phi_{xx}\phi_{xyy} - \phi_{yy}\phi_{xx}\phi_{xy} - \phi_y\phi_x\phi_{xxyy}&\\ +\phi_y\phi_x(\phi_x u_y + \phi_y u_x) + \phi_{yy}\phi_x\phi_{xxy} = 0.&\end{aligned} \tag{46}$$

Because of the conformal invariance of the Schwartz BLMP equation (45) and the transformation relation (46), we find that the conformal flow of the BLMP equation has the form

$$u_t = -2\phi_{xy} \tag{47}$$

with (46).

Making the transformation

$$\phi_x = \exp w \tag{48}$$

to equation (46), the corresponding $(2+1)$-dimensional ShG equation can be written in the form:

$$u_t = -2w_y \exp w \tag{49}$$

$$w_y = r_x + r w_x \tag{50}$$

$$(\partial_y + r_x\partial_x - r_{xy} - r r_{xx} + 2r_x^2)\left(\frac{r_x}{r}(w_{xy} - u)\right) = r_x^2(w_{xxy} - w_x w_{xy}). \tag{51}$$

Similarly, the conformal flow equation of the NNV equation (44) possesses the form

$$u_t = -2\phi_{xy} \tag{52}$$

with

$$\begin{aligned}
u(2\phi_y^4\phi_x\phi_{xy} + 2\phi_x^4\phi_y\phi_{xy} - \phi_y^3\phi_x^2\phi_{yy} - \phi_y^2\phi_x^3\phi_{xx} - \phi_y^5\phi_{xx} - \phi_x^5\phi_{yy})\\
+u_x(\phi_x^4\phi_y^2 + \phi_y^5\phi_x) + u_y(\phi_y^4\phi_x^2 + \phi_x^5\phi_y)\\
-\phi_{xx}\phi_{yy}\phi_{xy}(\phi_x^3 + \phi_y^3) + \phi_y\phi_x(\phi_y^2\phi_{yy}\phi_{xxy} + \phi_x^2\phi_{xyy}\phi_{xx})\\
-\phi_y\phi_x(\phi_y^3 + \phi_x^3)\phi_{yyxx} + \phi_y^4\phi_{xx}\phi_{xyy} + \phi_x^4\phi_{xxy}\phi_{yy} = 0.
\end{aligned} \tag{53}$$

One can easily verify that when $y = x$, equation (48) transforms (53) to the usual Miura transformation and (52) to the known $(1+1)$-dimensional ShG equation. So the conformal flow equation (52) with (53) is a variant form of an integrable $(2+1)$-dimensional ShG extension related to the NNV equation.

In summary, if we have a $(1+1)$- or $(2+1)$-dimensional extension of the KdV equation, we can also obtain a $(1+1)$- or $(2+1)$-dimensional ShG (or SG) model from the conformal flow equation of the KdV extension. The ShG (or SG) extensions can be obtained by means of the generalized Miura transformations and the conformal flow equations of the KdV equations. The conformal flow equations can be obtained from the standard singularity analysis and the Möbious transformation invariance of the singular manifold equations of the KdV equations.

4. Integrable models from the conformal constraints

To find some integrable models (especially in higher dimensions) is also an important topic in nonlinear science. The symmetry constraint (and/or reduction) method is one of the most powerful tools to give new integrable models from known ones. In this section, we use the symmetry constraints of the KP equation related to the conformal invariance to obtain the corresponding integrable models both in $(1+1)$ and $(2+1)$ dimensions.

4.1. (1 + 1)-dimensional constraints of the KP equation

In principle, every one symmetry of a higher-dimensional model can be used to reduce the original model to its lower form. For instance, the symmetry constraint conditions

$$u_y = 0 \qquad u_t = 0 \qquad u_t = 0 \tag{54}$$

which are corresponding to the spacetime translation invariance will reduce the KP equation (33) to the KdV equation

$$u_t + u_{xxx} + 6uu_x = 0 \tag{55}$$

the Bousinesq equation

$$u_{xxxx} + 6uu_{xx} + 6u_x^2 + 3u_{yy} = 0 \tag{56}$$

and the linear wave equation

$$u_{yy} = 0 \tag{57}$$

respectively. The most general Lie point symmetry constraints and the conditional Lie point symmetry constraints reduce the KP equation (33) to the same equations (54)–(57) but with different independent arguments [13, 14]. Now we use the conformal constraints to find some more integrable models from the KP equation.

Let ψ_i and ψ_j^*, $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$, be independent solutions of the Lax pairs

$$\psi_{xx} - u\psi + \psi_y \equiv L_1\psi = 0 \tag{58}$$

$$\psi_t + 4\psi_{xxx} - 6u\psi_x - 3\left(u_x - \int u_y\,\mathrm{d}x\right)\psi \equiv L_2\psi = 0 \tag{59}$$

and

$$\psi^*_{xx} - u\psi^* - \psi^*_y \equiv L_1^*\psi^* = 0 \tag{60}$$

$$\psi^*_t + 4\psi^*_{xxx} - 6u\psi^*_x - 3\left(u_x + \int u_y\,\mathrm{d}x\right)\psi^* \equiv L_2^*\psi^* = 0. \tag{61}$$

Then from the last section we know that $(\psi_i\psi_j^*)_x$ for all i, j are conformal invariance related symmetries of the KP equation. Now substituting the symmetry constraint condition

$$u_x = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}(\psi_i\psi_j^*)_x \tag{62}$$

where a_{ij} are arbitrary constants, into equations (58) and (60) we get a generalized $(N+M)$-component AKNS system:

$$\psi_{iy} = -\psi_{ixx} + \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_n\psi_m^*\psi_i \qquad i = 1, 2, \ldots, N \tag{63}$$

$$\psi^*_{jy} = \psi^*_{jxx} - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_n\psi_m^*\psi_j^* \qquad j = 1, 2, \ldots, M. \tag{64}$$

When $M = N$

$$a_{nm} = \delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

the equation system (63) and (64) is reduced to the usual $2N$-component AKNS system [10].

Substituting (62) into (59) and (61) and using (63) and (64), the t-part of the Lax pair becomes a generalized $(N+M)$-component modified KdV (mKdV) system ($i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$):

$$\psi_{it} = -4\psi_{ixxx} + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_n\psi_m^*\psi_{ix} + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_{nx}\psi_m^*\psi_i \tag{65}$$

$$\psi^*_{jt} = -4\psi^*_{jxxx} + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_n\psi_m^*\psi^*_{jx} + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_n\psi^*_{mx}\psi_j^*. \tag{66}$$

When $M = N$, $a_{nm} = \delta_{nm}$, the equation system (65) and (66) is reduced to the usual $2N$-component mKdV system [15]. More especially, when $N = M = 1$, $\psi_1 = \psi$, $\psi_1^* = \psi^*$, $a_{11} = 1$, equation system (63) and (64) is the usual nonlinear Schrödinger (NLS) equation

$$\psi_y = -\psi_{xx} + \psi^2\psi^* \tag{67}$$

$$\psi^*_y = \psi^*_{xx} - \psi\psi^{*2} \tag{68}$$

and the system (65) and (66) is just the mKdV equation,

$$\psi_t = -4\psi_{xxx} + 12\psi\psi^*\psi_x \tag{69}$$

$$\psi^*_t = -4\psi^*_{xxx} + 12\psi\psi^*\psi^*_x. \tag{70}$$

4.2. (2 + 1)-dimensional constraints of the KP equation

It is obvious that the KP equation is invariant under the inner parameter translation, say z translation. That is to say q_z is also a symmetry of the KP equation. So similar to $(1+1)$-dimensional constraint condition (62), we can use

$$q_z = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}(\psi_i\psi_j^*)_x \tag{71}$$

where $a_{ij},\ i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M$ are constants, ψ_i and ψ_j^* are independent solutions of the Lax pairs (58) and (59) and (60) and (61), as a new conformal constraint to obtain a generalized $(2+1)$-dimensional AKNS system. Substituting the constraint condition (71) to (58) and (60) for $\psi = \psi_i,\ \psi^* = \psi_j^*$, we find a generalized $(N + M)$-component $(2 + 1)$-dimensional AKNS extension:

$$\psi_{iy} = -\psi_{ixx} + \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_i\partial_z^{-1}(\psi_n\psi_m^*)_x \qquad i = 1, 2, \ldots, N \tag{72}$$

$$\psi_{jy}^* = \psi_{jxx}^* - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}\psi_j^*\partial_z^{-1}(\psi_n\psi_m^*)_x \qquad j = 1, 2, \ldots, M. \tag{73}$$

When we take $M = N = a_{11} = 1,\ \psi_1 = \psi,\ \psi_1^* = \psi^*$, we get the simplest special case of (72) and (73):

$$\psi_y = -\psi_{xx} + \psi\partial_z^{-1}(\psi\psi^*)_x \tag{74}$$

$$\psi_y^* = \psi_{xx}^* - \psi^*\partial_z^{-1}(\psi\psi^*)_x. \tag{75}$$

One can easily prove that the equation system (74) and (75) is a flow equation of the well known DS equation [16]:

$$\psi_y = -\psi_{xx} - \psi_{zz} + \psi\partial_z^{-1}(\psi\psi^*)_x + \psi\partial_x^{-1}(\psi\psi^*)_z \tag{76}$$

$$\psi_y^* = \psi_{xx}^* + \psi_{zz}^* - \psi^*\partial_z^{-1}(\psi\psi^*)_x - \psi^*\partial_x^{-1}(\psi\psi^*)_z. \tag{77}$$

In fact the system (74) and (75) is a space $\{x, z\}$ asymmetric form of the DS equation. So we call (74) and (75) the asymmetric DS (ADS) system and the system (72) and (73) $(N + M)$-component ADS system.

Substituting (71) into (59) and (61) and using (72) and (73), the t-part of the Lax pair becomes the generalized $(N + M)$-component $(2 + 1)$-dimensional modified KdV (mKdV) system ($i = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M$):

$$\psi_{it} = -4\psi_{ixxx} + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}(\psi_{ix}\partial_z^{-1}(\psi_n\psi_m^*)_x + \psi_i\partial_z^{-1}(\psi_{nx}\psi_m^*)_x) \tag{78}$$

$$\psi_{jt}^* = -4\psi_{jxxx}^* + 6\sum_{n=1}^{N}\sum_{m=1}^{M} a_{nm}(\psi_{jx}^*\partial_z^{-1}(\psi_n\psi_m^*)_x + \psi_j^*\partial_z^{-1}(\psi_n\psi_{mx}^*)_x). \tag{79}$$

Taking $M = N = a_{11} = 1,\ \psi_1 = \psi,\ \psi_1^* = \psi^*$, the equation system (78) and (79) becomes

$$\psi_t = -4\psi_{xxx} + 6\psi_x\partial_z^{-1}(\psi_n\psi_m^*)_x + 6\psi\partial_z^{-1}(\psi_x\psi^*)_x \tag{80}$$

$$\psi_t^* = -4\psi_{xxx}^* + 6\psi_x^*\partial_z^{-1}(\psi\psi^*)_x + 6\psi^*\partial_z^{-1}(\psi\psi_x^*)_x. \tag{81}$$

The system (80) and (81) reduces to the known asymmetric NNV (ANNV) equation,

$$\psi_t = -4\psi_{xxx} + 3(\psi\partial_z^{-1}\psi)_x \tag{82}$$

for $\psi^* = 1/2$. For $\psi^* = \psi$ and $z = x$, the system (80) and (81) becomes the modified KdV equation. So we call system (80) and (81) the modified ANNV equations and system (78) and (79) the $(M + N)$-component modified ANNV system.

Using the similar conformal constraints to other integrable models we may get many new integrable models, but here we do not discuss this problem further.

5. Summary and discussion

Conformal invariance plays a very important role in many physics fields. In particular, in integrable theory, using the Painlevé analysis, most integrable models can be changed to their conformal invariant forms (Schwartz forms). Starting from the conformal invariance of the Schwartz forms of integrable models, various other interesting properties can be obtained at the same time. For instance, using the conformal invariance of an integrable model, say, KdV equation, we may find infinitely many symmetries. For the KdV equation, we have obtained one set of infinitely many local and three sets of infinitely many non-local symmetries from the conformal invariance.

For every kind of integrable extension of the KdV equation, there must exist a ShG extension. The ShG extensions can be obtained from the conformal flow equations of the corresponding KdV extensions and generalized Miura transformations. In this paper, the concrete ShG extensions related to the Hirota–Satsuma, the Kadomtsev–Petviashvili, the Boiti–Leon–Manna–Pempinelli and the Nizhnik–Novikov–Veselov equations are given.

Using every symmetry constraint to a higher-dimensional integrable model, one can obtain a lower-dimensional integrable model. Using a conformal invariance related symmetry constraint to the KP equation, a generalized $(N + M)$-component AKNS system is obtained. A more interesting result is that we can embed a lower-dimensional integrable model to higher dimensions. Then using the symmetry constraints in the enlarged spacetime, we can also obtain higher-dimensional integrable models. For the KP equation, after embedding it to $(3 + 1)$ dimensions $\{x, y, z, t\}$ (i.e. considering that the solution of the KP equation u is not only a function of the explicit space variables x, y and t but also a function of an inner space variable z), some $(2+1)$-dimensional integrable models including the asymmetric Davey–Stewartson equation and asymmetric Nizhnik–Novikov–Veselov (or named BLMP) equation are obtained from the symmetry constraints related to the conformal invariance and the inner-space translation invariance.

In summary, when a model possesses a conformal invariant form, many interesting integrable properties, such as infinitely many symmetries, and conservation laws can be obtained. Furthermore, starting from the conformal invariance of an integrable model, one can also find many new integrable models from the conformal flows and conformal constraints.

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